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Editorial: Hierarchical and Bilevel Programming

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Abstract. Approximately twenty years ago the modern interest for hierarchical programming was initiated by J. Bracken and J.M. McGill [9], [10]. The activities in the field have ever grown lively, both in terms of theoretical developments and terms of the diversity of the applications. The collection of seven papers in this issue covers a diverse number of topics and provides a good picture of recent research activities in the field of bilevel and hierarchical programming. The papers can be roughly divided into three categories; Linear bilevel programming is addressed in the first two papers by Gendreau et al and Moshirvaziri et al; The following three papers by Nicholls, Loridan & Morgan, and Kalashnikov & Kalashnikova are concerned with nonlinear bilevel programming; and, finally, Wen & Lin and Nagase & Aiyoshi address hierarchical decision making issues relating to both biobjective and bilevel programming.

Keywords: Hierarchical Programming, Bilevel Programming, Global Optimization, Multiobjective Programming, Stackelberg Game

1. Introduction

Hierarchical programming is concerned with decision making problems that involve multiple decision makers ordered within a hierarchical structure. The higher order decision makers strongly influence the decisions of those with a lower rank. The most well-known case, described by the so-called Stackelberg game [25], [26], [1], [5], is the one in which decision makers of two different ranks are involved. This is the bilevel case. In fact, there has been little work done on the general multilevel hierarchical problem [2], [27]. Considering the complexity of the problem [14], [8], [27], even for the linear bilevel case, this should not surprise.

2. The Bilevel Programming Problem

The bilevel programming problem is formulated as follows:

$$(\mathbf{P}_1) \qquad \min_{\mathbf{y}\in\mathcal{Y}} f_1(\mathbf{x}(\mathbf{y}),\mathbf{y}) \tag{1}$$

$$\text{s.t.}\,\mathbf{g}_1(\mathbf{x}(\mathbf{y}),\mathbf{y}) \leq \mathbf{0} \tag{2}$$

$$\text{where } (\mathbf{P}_2) \quad \mathbf{x}(\mathbf{y}) \in \arg\min_{\mathbf{x}\in\mathcal{X}} f_2(\mathbf{x},\mathbf{y}) \tag{3}$$

$$\text{s.t.} \mathbf{g}_2(\mathbf{x},\mathbf{y}) \leq \mathbf{0} \tag{4}$$

$\mathbf{P}_1$ is called the *upper* or *first level* problem, while $\mathbf{P}_2$ is the *lower* or *second level* problem.

The *linear bilevel problem*, i.e., the case where all functions involved are affine, is usually stated as follows:

$$(\mathbf{LP}_1) \qquad \min_{\mathbf{y}} \mathbf{c}_1^T\mathbf{x} + \mathbf{d}_1^T\mathbf{y} \tag{5}$$

$$\text{where } \mathbf{x} \text{ solves } (\mathbf{LP}_2) \quad \min_{\mathbf{x}} \mathbf{c}_2^T\mathbf{x} + \mathbf{d}_2^T\mathbf{y} \tag{6}$$

$$\text{s.t.} \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{y} \leq \mathbf{b} \tag{7}$$

$$\mathbf{x},\mathbf{y} \geq \mathbf{0} \tag{8}$$

The linear case and the nonlinear case with (strictly) convex second level problem have attracted the greatest attention in the literature [2], [26], [20], [23].

Despite its designation as "linear", problem $(\mathbf{LP}_1)$-$(\mathbf{LP}_2)$ is *NP-hard* and belongs to the realm of global optimization [8], [14], [4], [12], [6].

In the paper contributed by M. Gendreau, P. Marcotte and G. Savard, an adaptive search method related to the Tabu metaheuristic is developed for the linear bilevel problem and large problem instances are solved with good accuracy and low CPU times.

The paper contributed by K. Moshirvaziri, M. Amouzegar and S. Jacobsen presents an easy-to-implement method of constructing test problems for linear bilevel programming problems. It requires only the use of linear programming and local vertex enumeration.

In the paper contributed by M. Nicholls, a special case of the nonlinear bilevel programming problem is treated. It models the complete operation of the Portland Aluminium Smelter in Victoria, Australia. The aim is to maximize the aluminium production while minimizing the main costs associated with the production. A solution approach based on local vertex enumeration and grid search tailored to the particular problem structure is developed.

2.1. The Stackelberg Duopoly

In this case, the bilevel programming problem describes a hierarchical system that is composed of two levels of decision makers. The higher level decision maker, known as *leader*, controls the decision variables $\mathbf{y} \in \mathcal{Y}$, while the lower level decision maker, known as *follower*, controls the decision variables $\mathbf{x} \in \mathcal{X}$. The interaction between the two levels is modelled in their respective *loss functions* $f_1(\mathbf{x},\mathbf{y})$ and $f_2(\mathbf{x},\mathbf{y})$. The idea of the *Stackelberg duopoly game* [25], [1], [5] is as follows: The first player, the leader, chooses $\mathbf{y} \in \mathcal{Y}$ to minimize the loss function $f_1(\mathbf{x},\mathbf{y})$, while the second player, the follower reacts to leader's decision by selecting a strategy $\mathbf{x} \in \mathcal{X}$ that minimizes his loss function $f_2(\mathbf{x}, \mathbf{y})$, in full knowledge of the leader's decision. Thus, the follower's decision depends upon the leader's decision, i.e. $\mathbf{x} = \mathbf{x}(\mathbf{y})$, and the leader is in full knowledge of this.

Consequently, we have the following definition:

Definition. If there exists a mapping $\mathbf{x} : \mathcal{Y} \to \mathbf{X}$ such that for any fixed $\mathbf{y} \in \mathcal{Y}$,

$$f_2(\mathbf{x}(\mathbf{y}), \mathbf{y}) \ \leq \ f_2(\mathbf{x}, \mathbf{y}), \ \ \forall \mathbf{x} \in \mathcal{X}, \tag{9}$$

and if there exists $\mathbf{y}^\star \in \mathcal{Y}$ such that

$$f_1(\mathbf{x}(\mathbf{y}^\star), \mathbf{y}^\star) \ \leq \ f_2(\mathbf{x}(\mathbf{y}), \mathbf{y}), \ \ \forall \mathbf{y} \in \mathcal{Y}, \tag{10}$$

then the pair $(\mathbf{x}^\star, \mathbf{y}^\star)$, where $\mathbf{x}^\star = \mathbf{x}(\mathbf{y}^\star)$, is called a *Stackelberg equilibrium* with the first player as leader and the second player as follower.

According to the definition, the Stackelberg equilibrium prescribes an optimal strategy for the leader if the follower reacts by playing optimally, whenever the leader announces his move first.

The bilevel programming model that corresponds to the conditions (9) -(10) of the definition above is:

$$\min_{\mathbf{y} \in \mathcal{Y}} \quad f_1(\mathbf{x}(\mathbf{y}), \mathbf{y}) \tag{11}$$

$$\text{where } \ \mathbf{x}(\mathbf{y}) \ = \ \arg\min_{\mathbf{x} \in \mathcal{X}} f_2(\mathbf{x}, \mathbf{y}), \tag{12}$$

where we have used implicitly the assumption that [5], [17] *the follower's response to every strategy of the leader is unique*, i.e., the reaction set of the follower, $\mathcal{X}^\star(\mathbf{y}) = \arg\min_{\mathbf{x} \in \mathcal{X}} f_2(\mathbf{x}, \mathbf{y})$ is always a singleton.

Whenever this assumption is not satisfied, there may be ambiguity in the possible responses of the follower and consequently in the attainable loss levels of the leader. If $\mathcal{X}^\star(\mathbf{y})$ is not always a singleton, the Stackelberg solution concept introduced in the definition above is not directly applicable.

One way to remove the ambiguity in the attainable loss levels of the leader would be to stipulate that the leader's attitude is towards minimizing the worst outcome rather than towards taking risks. This introduces the *weak Stackelberg problem*[5], [17] :

$$\min_{\mathbf{y} \in \mathcal{Y}} \quad \{ \max_{\mathbf{x} \in \mathcal{X}^\star(\mathbf{y})} f_1(\mathbf{x}, \mathbf{y}) \} \tag{13}$$

$$\text{where } \ \mathcal{X}^\star(\mathbf{y}) \ = \ \arg\min_{\mathbf{x} \in \mathcal{X}} f_2(\mathbf{x}, \mathbf{y}), \tag{14}$$

The corresponding *strong Stackelberg problem* [16], [17] would then be

$$\min_{\mathbf{y}\in\mathcal{Y}} \quad \{\min_{\mathbf{x}\in\mathcal{X}^\star(\mathbf{y})} f_1(\mathbf{x},\mathbf{y})\} \tag{15}$$

$$\text{where } \mathcal{X}^\star(\mathbf{y}) = \arg\min_{\mathbf{x}\in\mathcal{X}} f_2(\mathbf{x},\mathbf{y}), \tag{16}$$

In the paper contributed by P. Loridan and J. Morgan, the method of Molodtsov[22] is used to approximate the weak Stackelberg problem with a sequence of strong Stackelberg problems.

2.2. Minimizing over a Noncooperative Equilibrium

An interesting case of the bilevel programming problem arises when the second level consists of more than one antagonistic decision makers, all involved in a Nash game. Such problems frequently arise in mixed economies, land-use and traffic planning [2], [18], [20]. They are often formulated as mathematical programming problems with variational inequality constraints, thus:

$$\min \quad f_1(\mathbf{x},\mathbf{y}) \tag{17}$$

$$\text{s.t.} \quad \mathbf{y}\in\mathcal{Y} \tag{18}$$

$$\mathbf{g}(\mathbf{x},\mathbf{y})^T(\mathbf{z}-\mathbf{x}) \geq 0, \forall \mathbf{z}\in\mathcal{X}(\mathbf{y}) \tag{19}$$

$$\mathbf{x}\in\mathcal{X}(\mathbf{y}) \tag{20}$$

Problems with variational inequality constraints arise also in the context of shape and structure optimization as well as in other engineering applications [24], [11], [13], [26].

In the note contributed by V. Kalashnikov and N. Kalashnikova, the related problem of finding a vector $\mathbf{y}^\star \in \mathcal{Y} = \{\mathbf{y}\in\mathcal{X} : \mathbf{g}(\mathbf{y})^T(\mathbf{x}-\mathbf{y}) \geq 0, \forall \mathbf{x}\in\mathcal{X}\}$ such that $\mathbf{f}(\mathbf{y}^\star)^T(\mathbf{y}-\mathbf{y}^\star) \geq 0, \forall \mathbf{y}\in\mathcal{Y}$, is considered and a penalty function method based on parametric variational inequalities is developed for the solution of the problem under suitable assumptions.

3. Relations to Multiobjective Programming and Optimization over Efficient Sets

The relations of the bilevel optimization problem (11)-(12) to the bicreteria optimization problem

$$\min \quad \mathbf{f}(\mathbf{z}) \tag{21}$$

$$\text{s.t.} \quad \mathbf{z}\in\mathcal{Z}, \tag{22}$$

where $\mathbf{z} = (\mathbf{x}, \mathbf{y})$, $\mathcal{Z} = \mathcal{X} \times \mathcal{Y}$, and $\mathbf{f} = (f_1, f_2)^T$, has been debated in the past. In particular, the question whether the bilevel optimization problem can be solved as an equivalent bicriteria optimization problem has been addressed by several authors. Such an approach has been taken previously by [3], [28] in the linear case and by [15] for a special nonlinear problem. However, although such an approach can be justified in some special cases [21], it is generally inadequate as demonstrated with the means of counter examples in [6], [29], [19]. Thus, the solution to the bilevel problem may not be efficient.

On the other hand, if both levels agree that the result of such a system is economically inadmissible, they may be willing, if allowed, to cooperate when the optimal solution has been found inefficient. In the paper contributed by U.-P. Weng and S.-F. Lin, the notion of *cooperative bilevel programming problem* is introduced for this purpose. Its characteristics are studied and an efficient approach based on goal settings by the decision makers is proposed for its solution.

The paper contributed by K. Nagase and E. Aiyoshi is concerned with the theme of how to choose the best solution out of a set of non-inferior solutions to a bicreteria optimization problem. For this purpose, the bilevel optimization problem

$$\min_{\mathbf{w}} \varphi(\mathbf{f}(\mathbf{w})) \tag{23}$$
$$\text{s.t.} \quad \mathbf{w} \in \arg\min_{\mathbf{z} \in \mathcal{Z}} \mathbf{f}(\mathbf{z}) \tag{24}$$

is formulated. Here the first level is a preference optimization problem and the second level is a bicreteria optimization problem. This kind of hierarchical problems, known as *optimization over efficient sets*, have previously been analysed by [7] for the linear case.

The first level objective function, φ, is a social preference constructed from the individual preferences of the decision makers using the simple majority decision rule. Under proper assumptions, a solution to (23)-(24) is obtained by transforming it to an ε-parameter choice problem using the ε-constraint method, and applying the golden section method.

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